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MODELING AND NONLINEAR OPTIMAL CONTROL OF N-ROTOR VTOL UNMANNED AERIAL VEHICLES

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Abstract— Quadcopters, hexa-copters and multi-rotor unmanned aerial vehicles (UAV) in general have become one of the most common types of vertical take-off and landing (VTOL) aerial vehicles where the thrust vectors of all the rotors are usually parallel. These UAVs are typically under-actuated meaning that the number of actuators is less than the degrees of freedom of the vehicle and because of that they cannot achieve holonomic motion. Recently, new designs for multi-rotor UAVs are proposed where the thrust vectors of the rotors are not necessarily parallel, and the rotors can have specific orientations with respect to the body of the vehicle. These new designs can achieve holonomy by manipulating thrust forces and moments of individual rotors which results in independent control of attitude and position of the UAV. In this paper, modeling of forces and moments of rotors is presented first. Second, translation and rotation model of the vehicle is presented, followed by nonlinear optimal control design for attitude and position of the vehicle. Attitude controller is designed according to the state dependent Riccati equation (SDRE) in nonlinear optimal control. Using input-state feedback linearization technique, we simplify the problem and then an analytical solution - utilizing quaternion parameters - for the SDRE is presented. Similarly, a linear quadratic regulator (LQR) for controlling the speed and position of the vehicle is designed. In addition, using Lyapunov theory, proofs for global asymptotic stability of all controllers are provided. Finally, simulations verifying the results are presented.

Keywords; UAV; VTOL; Optimal Control; SDRE; LQR

I. INTRODUCTION

Vertical Take-Off and Landing (VTOL) unmanned aerial vehicles (UAVs) have received significant attention both in academia and industry in recent years. Low maintenance and production cost, simplicity and maneuverability are of the few reasons why these vehicles have become so popular in both research and industry area. Multi-rotors (including quadcopters and hexa-copters) are one of the most common type of these UAVs where they have individual rotors generating thrust forces and moments with their parallel thrust force vectors. These UAVs are usually under-actuated meaning that only a subset of all degrees of freedom can be independently manipulated. In multi-rotors usually altitude and attitude degrees of freedom are controlled independently and the remaining two translational degrees of freedom are controlled indirectly by manipulating the attitude of the vehicle [1, 2, 3]. For such under-actuated systems, usually a cascaded control strategy is developed where the output of the inner control loop is the input of the outer control loop.

Although during the past decades different control algorithms have been developed to cope with the limitations of actuators and improving flight performance [3, 4, 5], they still cannot achieve holonomy using the common under-actuated multi-rotor UAV design. To solve this issue, multi-rotor UAVs with tilting rotors are proposed [6, 7]. Multi-rotors with fixed but tilted rotors are proposed in [8] where the thrust force vectors of all rotors are not parallel. In [9, 10], a new design is introduced where the rotors could tilt on the fly. The work in [11] uses ducted fans to design a holonomic UAV. Also, a quadcopter with tilting rotors is presented in [12]. The work in [13] shows that using a single central motor and 3 tilting rotors holonomic motion can be achieved. Similarly, the method in [14] uses two counter rotating central rotors in addition to three tilting ducted fans to achieve holonomy.

Although papers in the literature addressed under-actuation problem of VTOL UAVs, their modeling and control methods are limited to the specific cases with the specified number of rotors and tilting angles. Therefore, the general formulation for both modeling and control parts which is not dependent on the specific type or number of rotors is still remaining. This paper addresses this issue and presents the first general modeling and control method for multi-rotors. In this way, Newton-Euler method and some novel nonlinear optimal controllers with stability proof are used for modeling and control parts of the UAV, respectively.

The rest of this paper is organized as follows. The general nonlinear dynamic model of multi-rotor is presented in section II. Sections III and IV include details of the control design process and simulation results, respectively. Finally, section V concludes the work.

II. Modeling

To derive a 6 degree-of-freedom (DOF) equation of motion, two reference frames should be defined. First, an inertial reference frame denotes $\mathcal{F}_I$ which is attached to the earth surface. Second, body reference frame which is indicated by $\mathcal{F}_B$ and is fixed to the center of gravity of the aerial vehicle. Six DOF motion can be divided into two separate translational and rotational motion. Therefore, the equation of each motion is derived based on Newton-Euler method in the following sections.

A. Rotational Motion

Rotational motion equation for the multirotor can be written as

$$\sum M = \frac{dH}{dt}\Big|_I = \frac{dH_B}{dt}\Big|_I + \sum_{i=1}^{4} \frac{dH_r^i}{dt}\Big|_I, \tag{1}$$

where $\sum M$ is the total external torques implied on the aerial vehicle and H, H_B and H_r^i are total angular momentum of the system, angular momentum of the UAV body and total angular momentum of motors and their propellers, respectively. All derivatives in (1) are derived with respect to $\mathcal{F}_I$ frame. Using the derivation relationship in rotational frames, (1) can be rewritten as

$$\sum M = \frac{dH_B}{dt}\Big|_B + \sum_{i=1}^{4} \frac{dH_r^i}{dt}\Big|_B + S_\omega\left(H_B + \sum_{i=1}^{4} H_r^i\right) = J_B\dot{\omega} + S_\omega J_B \omega + \sum_{i=1}^{4} \dot{H}_r^i + \sum_{i=1}^{4} S_\omega\ H_r^i, \tag{2}$$

where J_B is the moment of inertia of UAV in $\mathcal{F}_B$ frame and remains constant in this frame. Vector $\omega = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ is the angular velocity of UAV at $\mathcal{F}_I$, $\dot{\omega}$ is the time derivative of ω and S_ω is the skew symmetric matrix of ω vector defined as

$$S_\omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{3}$$

After some simplification of (2), $\dot{\omega}$ can be written as

$$\dot{\omega} = J_B^{-1}\left(\sum M - S_\omega J_B \omega - \sum_{i=1}^{4} \dot{H}_r^i - \sum_{i=1}^{4} S_\omega\ H_r^i\right). \tag{4}$$

To calculate $\sum M$ the governing equations of motors and propellers are needed to be derived as (5).

$$D_{prop}^i = \tau_{stator}^i - \dot{h}_r^i, \tag{5}$$

where D_{prop}^i is aerodynamical torque implied on each propeller while τ_{stator}^i and $\dot{h}_r^i$ are a torque generated by each brushless motor and a rate of angular momentum change for each set of motor and propeller, respectively. $\dot{h}_r^i$ can be defined as (6).

$$\dot{h}_r^i = J_r^i \dot{\omega}_r^i. \tag{6}$$

Using (5), (4) can be rewritten as

$$\dot{\omega} = J_B^{-1}\left(\sum U(4{:}6) - S_\omega J_B \omega - \sum_{i=1}^{4} S_\omega\ H_r^i\right). \tag{7}$$

where $U(4{:}6)$ denotes the control input for attitude and is derived in section C. To derive rotational kinematics, quaternion parameters are used. The quaternion attitude equation set can be written as (8).

$$\dot{\epsilon} = \frac{1}{2}(\eta\omega + S_\epsilon \omega),$$

$$\dot{\eta} = -\frac{1}{2}\omega^T \epsilon, \tag{8}$$

where $\epsilon = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3]^T$ is the vector part of quaternion parameters, while η is the scalar part of that, and S_ϵ is the skew symmetric matrix of ϵ.

B. Tansitional motion

Using Newton's second law, translational equation of motion for UAV can be written as (9).

$$\frac{dV}{dt}\Big|_I = \frac{\sum F}{m},$$

$$\frac{dP}{dt}\Big|_I = V, \tag{9}$$

where $\sum F$ is the total external force implied on UAV, whereas V and P,respectively, are velocity and the position of UAV's center of the gravity in inertial reference frame. External forces implied on the UAV include gravity and thrust of motors are presented as.

$$\sum F = G - f, \tag{10}$$

where $G = [0 \quad 0 \quad mg]^T$ is the gravity vector in inertial reference frame in which m is the mass of the UAV. In addition, vector f is the thrust control vector in inertia frame defined in (12). Therefore, the translational equation can be rewritten as:

$$\dot{V} = G - \frac{f}{m},$$

$$\dot{P} = V, \tag{11}$$

where f can be defined as (12):

$$f = C_I^B U(1{:}3),$$

$$C_I^B = \begin{bmatrix} \eta^2 + \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \eta\epsilon_3) & 2(\epsilon_1\epsilon_3 + \eta\epsilon_2) \\ 2(\epsilon_1\epsilon_2 + \eta\epsilon_3) & \eta^2 - \epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \eta\epsilon_1) \\ 2(\epsilon_1\epsilon_3 - \eta\epsilon_2) & 2(\epsilon_2\epsilon_3 + \eta\epsilon_1) & \eta^2 - \epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2 \end{bmatrix}, \tag{12}$$

Here vector $U(1{:}3)$ is a control vector for position which is determined by using the motors thrust in the body frame (in the next section) and C_I^B is the rotation matrix between inertia and body frame.

C. Motors and propellers model

To complete the dynamic modeling of the multirotor UAV, sets of motors and propellers should be modeled. Note that in (5) D_{prop}^i is an external torque implied on each propeller, while τ_{stator}^i which causes rotational motion of the UAV is implied on the center of gravity of the vehicle. For simplicity, instead of considering $\dot{h}_r^i$, a first order transfer function with the time constant of σ is considered as motor's model in (13), where M_C^i is assumed as control input.

$$D_{prop}^i(s) = \frac{M_C^i(s)}{1+\sigma s}. \tag{13}$$

Furthermore, $h_r^i(s)$ can be simply calculated as (14).

$$h_r^i(s) = \frac{D_{prop}^i(s)}{s}. \tag{14}$$

Transfer function in (14) can also be used for modeling the thrust of motors as follows.

$$T^i(s) = \frac{T_C^i(s)}{1+\sigma s}. \tag{15}$$

To calculate the relationship between the thrust of motors T and their produced torques $U(4:6)$ and forces $U(1:3)$, in the body reference frame, it is assumed that the n[th] motor is located at the front of the UAV on the x axis of the body frame. The other motors are located at angles θ, 2θ, 3θ, …, $(n-1)\,\theta$ with respect to the x axis. Angle θ depends on the number of motors (n) and is equal to $\theta = \frac{2\pi}{n}$. Figure 1 illustrates the configuration of motors in horizontal plane.

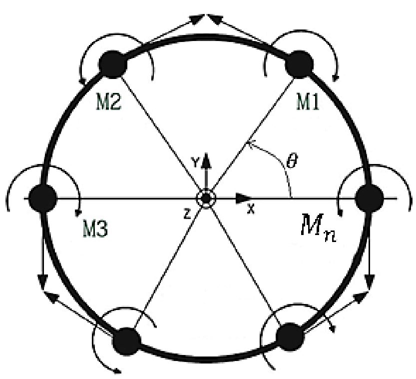

Figure 1. Motors arrangement in the horizontal plane

In the suggested arrangements of the motors in Figure 1, motors numbered with odd numbers rotate clockwise, whereas motors with even numbers rotate counterclockwise. Moreover, in the body frame's horizontal plane, thrusts of the odd motors are perpendicular to the radius of their located circles and they are deviated to the left side of their radiuses by the amount of installation angle (φ) in the vertical plane, while even motors are deviated to the right side.

Considering the suggested motors configuration, forces and torques resulted from even motors' thrust can be calculated in the body frame as (16). Note that F_f denotes the force generated by the motor while F_τ and τ_τ present the torque resulted from the motor's force and torque respectively.

$$\begin{bmatrix} F_{x_i} \\ F_{y_i} \\ F_{z_i} \\ \tau_{x_i} \\ \tau_{y_i} \\ \tau_{z_i} \end{bmatrix} = \begin{bmatrix} F_{f_{x_i}} \\ F_{f_{y_i}} \\ F_{f_{z_i}} \\ F_{\tau_{x_i}} + \tau_{\tau_{x_i}} \\ F_{\tau_{y_i}} + \tau_{\tau_{y_i}} \\ F_{\tau_{z_i}} + \tau_{\tau_{z_i}} \end{bmatrix} [T_i] = \begin{bmatrix} -\sin(\theta_i)\sin(\phi) \\ \cos(\theta_i)\sin(\phi) \\ \cos(\phi) \\ \sin(\theta_i)\,[d\cos(\phi) - k\sin(\phi)] \\ \cos(\theta_i)[-\cos(\phi) + k\sin(\phi)] \\ d\sin(\phi) + k\cos(\phi) \end{bmatrix} [T_i] = M_i T_i. \tag{16}$$

where d is the distance of each motor from the center of gravity of the UAV and T_i, $i = 1,2,\dots,n$ denotes the thrust of each motor. Also, k is the ratio of the torque to thrust produced by each motor. Similarly, for each odd motor one can write:

$$\begin{bmatrix} F_{x_i} \\ F_{y_i} \\ F_{z_i} \\ \tau_{x_i} \\ \tau_{y_i} \\ \tau_{z_i} \end{bmatrix} = \begin{bmatrix} F_{f_{x_i}} \\ F_{f_{y_i}} \\ F_{f_{z_i}} \\ F_{\tau_{x_i}} + \tau_{\tau_{x_i}} \\ F_{\tau_{y_i}} + \tau_{\tau_{y_i}} \\ F_{\tau_{z_i}} + \tau_{\tau_{z_i}} \end{bmatrix} [T_i] = \begin{bmatrix} \sin(\theta_i)\sin(\phi) \\ -\cos(\theta_i)\sin(\phi) \\ \cos(\phi) \\ \sin(\theta_i)\,[d\cos(\phi) - k\sin(\phi)] \\ \cos(\theta_i)[-\cos(\phi) + k\sin(\phi)] \\ -d\sin(\phi) - k\cos(\phi) \end{bmatrix} [T_i] \tag{17}$$

Therefore, total forces $U(1:3)$ and torques $U(4:6)$ produced by the motors can be derived in body frame as

$$U = \begin{bmatrix} F_x \\ F_y \\ F_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} F_{f_{x_1}} \\ F_{f_{y_1}} \\ F_{f_{z_1}} \\ F_{\tau_{x_1}} + \tau_{\tau_{x_1}} \\ F_{\tau_{y_1}} + \tau_{\tau_{y_1}} \\ F_{\tau_{z_1}} + \tau_{\tau_{z_1}} \end{bmatrix} + \cdots + \begin{bmatrix} F_{f_{x_n}} \\ F_{f_{y_n}} \\ F_{f_{z_n}} \\ F_{\tau_{x_n}} + \tau_{\tau_{x_n}} \\ F_{\tau_{y_n}} + \tau_{\tau_{y_n}} \\ F_{\tau_{z_n}} + \tau_{\tau_{z_n}} \end{bmatrix} = [M_1 \dots M_n] \begin{bmatrix} T_1 \\ . \\ . \\ . \\ T_n \end{bmatrix} = MT. \tag{18}$$

Equation (18) is invertible and therefore using the achieved control input from the controller, thrusts of the motors can be calculated as (19).

$$T = \begin{bmatrix} T_1 \\ . \\ . \\ . \\ T_n \end{bmatrix} = M^{-1} \begin{bmatrix} F_x \\ F_y \\ F_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = M^{-1}U. \tag{19}$$

III. CONTROL

In this section, control problem of rotational motion along with translational motion will be investigated.

A. *Attitude control*

Most previous works on attitude control of UAV adopted Euler angles as a feedback in control loop. This method needs some simplification to make the control process possible. Applying quaternion parameters, without any simplification, can lead to obtaining optimal attitude error in contrast with employing Euler angles. In addition, in control methods with quaternions, control law is a linear function of quaternion parameters. Therefore, working with quaternion is easier than Euler angles and adopting this easiness and simplicity, we suggest some analytical solutions, in this paper.

Another advantages of quaternions over Euler angles is that quaternions do not face singularity issue, while Euler angles may encounter singular points, and this can bring about some complexities in nonlinear control designs. In this paper, SDRE method is adopted for attitude control purpose. SDRE is a semi-optimal nonlinear control method and despite all advantages of this method, its stability analysis remains challenging. Although the global stability of such method is not determined, range of stability is estimated.

1) *Analytical solution for SDRE*

Generally, the input-state linearization method is used to design some nonlinear control systems. In these designs, all equations and state variables are commonly used, but here with the inspiration that comes from this technique, first, some nonlinear equation of motion for UAV are simplified. In the next step, simplified equations are written in the forms of SDRE method and subsequently, an analytical solution is suggested. Finally, by suggestion of Lyapunov candidate function, asymptotic stability of the closed loop system is proved.

a) Simplification of the attitude equations by input-state linearization method

Considering (7), control input υ can be redefined as

$$\upsilon = J_B^{-1}\left(\sum u - S_\omega J_B \omega - \sum_{i=1}^{4} S_\omega \ H_r^i\right), \tag{20}$$

and using this, the rotational dynamic of the system can be simplified as (21).

$$\dot{\omega} = \upsilon. \tag{21}$$

Therefore, control input u can be obtained from (20) as

$$u = J_B \upsilon + S_\omega J_B \omega + \sum_{i=1}^{4} S_\omega \ H_r^i. \tag{22}$$

b) Forming semi-linear state equation

Equations (8) and (21) form state equation of the system. UAV's angular velocity and vector part of quaternion parameters are set as states of the system. Therefore, on can write

$$x = \begin{bmatrix} \omega \\ \epsilon \end{bmatrix}, \ \dot{x} = f(x) + g(x)u$$

$$f(x) = \begin{bmatrix} 0 \\ \frac{1}{2}(\eta\omega + S_\epsilon \omega) \end{bmatrix}, g(x) = \begin{bmatrix} \acute{I} \\ 0 \end{bmatrix}, \tag{23}$$

where $\acute{I}$ is a 3 by 3 identity matrix, and $f(x)$ is a continuous function which $f(0) = 0$. Therefore, all conditions to form a semi-linear equation can be addressed as (24). That is:

$$\dot{x} = A(x)x + B(x)u, \ \begin{bmatrix} \dot{\omega} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_\epsilon & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \epsilon \end{bmatrix} + \begin{bmatrix} \acute{I} \\ 0 \end{bmatrix} \upsilon, \tag{24}$$

where A_ϵ is defined as:

$$A_\epsilon = \frac{1}{2} \begin{bmatrix} \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix}, \tag{25}$$

c) Closed solution for SDRE method

To solve the closed loop solution of SDRE method for attitude control, first, two positive definite diagonal weight matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ should be defined as

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

$$Q_1 = diag(q_{1i}^2), \ i = 1,2,3,$$

$$Q_2 = q_2^2 I,$$

$$R = r^2 I, \tag{26}$$

where r, q_2 and $q_{1i}, \forall i = 1,2,3$ are positive real numbers. Considering the response of Riccati equation in (27), as a positive definite matrix of P defined in (28), one can derive (29)-(32).

$$A^T P(x) + P(x)A(x) + Q(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0$$
(27)

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}. \tag{28}$$

$$A_\epsilon^T P_2 + P_2 A_\epsilon - \frac{1}{r^2} P_1^2 + Q_1 = 0, \tag{29}$$

$$A_\epsilon^T P_4 - \frac{1}{r^2} P_1 P_2 = 0, \tag{30}$$

$$P_4 A_\epsilon - \frac{1}{r^2} P_2^T P_1 = 0, \tag{31}$$

$$-\frac{1}{r^2} P_2^T P_2 + Q_2 = 0. \tag{32}$$

It should be noted that (30) and (31) are equivalent. Here, using (33), (24), (26) and input state linearization method, $\boldsymbol{\upsilon}$ can be obtained from (34).

$$u(x) = -K(x)x = -R^{-1}(x)B^T(x)P(x)x \tag{33}$$

$$\nu = -\frac{1}{r^2}(P_1\omega + P_2\epsilon). \tag{34}$$

Therefore, it is important to calculate matrices P_1 and P_2 by solving (29) and (32). Fist, equation (32) can be rewritten as:

$$P_2^T P_2 = r^2 Q_2. \tag{35}$$

This equation can be solved as

$$P_2 = \lambda r q_2 I. \tag{36}$$

Knowing P_2, one can obtain P_1 from (29) as

$$P_1^2 = r^2(A_\epsilon^T P_2 + P_2 A_\epsilon + Q_1). \tag{37}$$

Finally, knowing that P_1 is positive definite, one can solve (37) to find P_1 as (38)

$$P_1 = diag(r\sqrt{q_{1i}^2 + r q_2 \eta}) \ \ , i = 1, 2, 3 \tag{38}$$

d) Asymptotically global stability of the closed loop attitude control system

In this section, the global asymptotic stability of the attitude system is assessed using Lyapunov stability theory. Substituting (34) in (24), we have

$$\dot{\omega} = -\frac{1}{r^2}(P_1\omega + P_2\epsilon). \tag{39}$$

The following Lyapunov candidate is suggested for system in (39).

$$L = \frac{1}{2} r^2 \omega^T P_2^{-1} \omega + \epsilon^T \epsilon + (1 - \eta)^2, \tag{40}$$

where L is always positive ($L > 0$) and it is radially unbounded with respect to ω. Therefore, $\lim_{\omega \to \pm\infty} L = \infty$. The derivative of Lyapunov candidate can be attained as

$$\dot{L} = -\omega^T P_2^{-1} P_1 \omega. \tag{41}$$

It is obvious that

$$\begin{cases} \dot{L} < 0 \ \forall \omega \neq 0 \\ \dot{L} = 0 \ \forall \omega = 0 \end{cases}, \tag{42}$$

B. Translational velocity and position control

In this section, LQR method is adopted to design position and velocity control for multirotor UAV. Similar to the attitude control design analytical solution of LQR is obtained using input state linearization. Subsequently, we prove the stability of the closed-loop system by applying Lyapunov stability theory.

1) Analytical solution for LQR

Considering (11), new control input $\boldsymbol{z}$ can be selected as

$$z = G - \frac{f}{m}. \tag{43}$$

Therefore, translational dynamic equation can be simplified as

$$\dot{V} = z,$$
$$\dot{P} = V, \tag{44}$$

By finding $\boldsymbol{z}$ from LQR method, control input f can be easily calculated from (43) as

$$f = m(G - z). \tag{45}$$

a) Forming state equation

To calculate $\boldsymbol{z}$**,** velocity and position vectors are considered as state variables. Therefore, simplified state equations can be obtained in (46).

$$x = \begin{bmatrix} V \\ P \end{bmatrix}, \begin{bmatrix} \dot{V} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Í & 0 \end{bmatrix} \begin{bmatrix} V \\ P \end{bmatrix} + \begin{bmatrix} Í \\ 0 \end{bmatrix} z, \tag{46}$$

b) Closed solution for LQR method

Similar to the solution of the SDRE, First, two positive definite diagonal weight matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ as $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices for the position control are defined as

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$
$$M_1 = \mathrm{diag}(m_{1i}^2)\, Í, \tag{47}$$
$$M_2 = \mathrm{diag}(m_{2i}^2)\, Í,$$
$$N = \mathrm{diag}(n_i^2) Í,\ i = 1, 2, 3$$

where m_{1i}, m_{2i}, and n_i are positive real numbers. Considering positive definite matrix S in (48) as matrix P in (27), the response of Riccati equation can be found in (49)-(52).

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}, \tag{48}$$
$$S_2^T + S_2 - S_1 N^{-1} S_1 + M_1 = 0, \tag{49}$$
$$S_4 - S_1 N^{-1} S_2 = 0, \tag{50}$$
$$S_4 - S_2^T N^{-1} S_1 = 0, \tag{51}$$
$$-S_2^T N^{-1} S_2 + M_2 = 0. \tag{52}$$

Now, similar to the attitude control method, z can be calculated in (53).

$$z = -N^{-1}(S_1 \mathrm{V} + S_2 P), \tag{53}$$

Therefore, matrices S_1 and S_2 should be calculated from (49) and (52). To find S_2 (52) is rewritten as

$$S_2^T \sqrt{N^{-1}} \sqrt{N^{-1}} S_2 = (S_2^T \sqrt{N^{-1}})(S_2^T \sqrt{N^{-1}})^T = \sqrt{M_2}\sqrt{M_2}, \tag{54}$$

The solution of above equation is presented in (55).

$$S_2 = S_2^T = \zeta \sqrt{M_2}\sqrt{N}. \tag{55}$$

Knowing S_2, S_1 can be calculated by rewriting (49) as:

$$2\sqrt{M_2}\sqrt{N} + M_1 = S_1 N^{-1} S_1, \tag{56}$$

Finally, S_1 is obtained from (57).

$$S_1 = \sqrt{N}\sqrt{2\sqrt{M_2}\sqrt{N} + M_1}. \tag{57}$$

c) Asymptotically global stability of the closed loop attitude control system

Simplified close loop equation of motion can be noted as (58).

$$\dot{V} = -N^{-1}(S_1 V + S_2 P),$$
$$\dot{P} = V, \tag{58}$$

Now, the following Lyapunov candidate is suggested.

$$L = \frac{1}{2} V^T S_1^{-1} N V + \frac{1}{2} P^T S_1^{-1} S_2 P, \tag{59}$$

The above Lyapunov candidate is positive definite and is radially unbounded with respect to V and P. ($\lim_{V \to \pm\infty} L = \infty$, $\lim_{P \to \pm\infty} L = \infty$). The derivative of Lyapunov candidate can be attained from (60).

$$\dot{L} = -V^T V. \tag{60}$$

IV. Simulation Results

In this section, simulation results for tracking a desired helix path defined in (61) are presented. Desired Euler angles are considered zero. In this simulation, the simulated UAV equipped with 8 rotors having fixed mounted angles of 20 degree. Furthermore, initial Euler angles are assumed 5 degree, while other initial condition of the UAV states are considered zero. The characteristics of simulated UAV are shown in Table 1.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.5\cos\left(\frac{t}{2}\right) \\ 0.5\sin\left(\frac{t}{2}\right) \\ t/10 \end{bmatrix}, \tag{61}$$

Table **1** Specifications of simulated drone

Paremeter	**Descriptin**	**Value**
d	Distance between CG and each motor	0.21 m
m	UAV's mass	0.74 kg
J_B	UAV's moment of inertia matrix	$\begin{bmatrix} 0.004 & 0 & 0 \\ 0 & 0.004 & 0 \\ 0 & 0 & 0.008 \end{bmatrix}$ kg. m²
k	Proportion of torque over produced thrust	0.1

Figure 2 illustrates the 3-dimensional trajectory of the UAV whereas figure 3 shows the Euler angles of the simulated UAV. It is shown that how UAV's angles have changed from 5 degree to 0 and how UAV tracks the desired trajectory with zero desired Euler angles. The translational velocity of the UAV in three inertial axes and its norm are presented at Figure 4, while Figures 5 and 6 indicate the control results as produced thrust of each motor. As these graphs show, the thrust of the motors has oscillatory behavior in order to generate sufficient force and moments to track a desired helix trajectory with zero Euler angles.

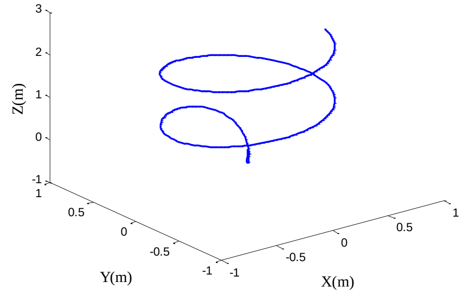

Figure 2. 3D trajectory of the UAV

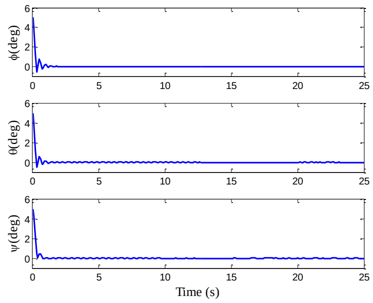

Figure 3. Euler angles of the UAV

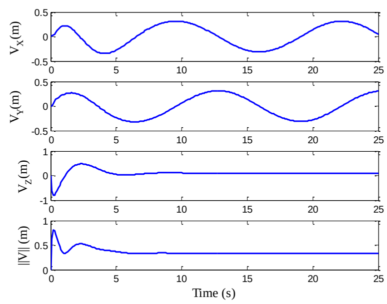

Figure 4. Velocity of the UAV

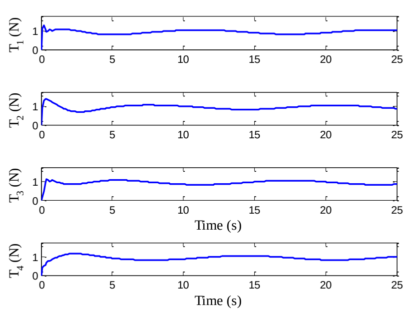

Figure 5. Trust produced by the UAV's motors (1 to 4)

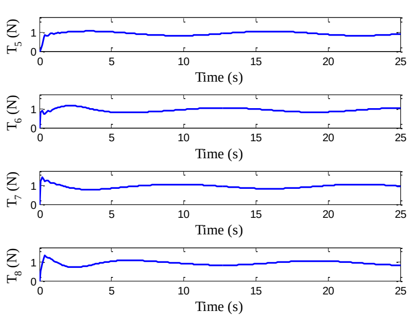

Figure 6. Peoduced thrust by the UAV's motors (5 to 8)

V. Conclusion

In this paper, modeling and control of the n-rotor VTOL UAV are presented. The suggested general formulation and control strategy are not dependent on the number of motors and their installation angles. Therefore, they can be used for a verity of VTOL's configurations. Furthermore, the proposed method makes it possible to control the attitude and position of the UAV, independently. It means that UAV can freely fly with 6 DOF controllability. Moreover, this paper proposed an optimal controller to control both attitude and position tasks. The stability of the proposed control schemes is also proved through Lyapunov method which makes the designed controllers reliable. To analyze the system's behavior and verify the ability of the proposed controllers, simulation results for tracking a helix trajectory are presented, demonstrating the performance of the UAV and suggested controllers.

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